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The construction of Kochen–Specker noncolourable sets in higher-dimensional space from corresponding sets in lower dimension: modification of Cabello, Estebaranz and García-Alcaine's method

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Abstract

By re-examination of Cabello, Estebaranz and García-Alcaine's method (CEG) for constructing a ray set which gives a proof of Kochen–Specker theorem in a higher dimension from an already known set in a lower dimension, a more refined method is derived. In the construction of ray sets of higher dimension, we need fewer rays than in CEG. By using the method, an analytical proof of the KS-noncolourability of ray sets in real Hilbert spaces is also given whose KS-noncolourability was found by CEG by computer calculation.

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1. Introduction

The Bell–Kochen–Specker (BKS) theorem [1, 2] is known as one of the theorems that state the difference between the notion of probability in quantum mechanics (QM) and classical probability theory. The theorem states that we cannot find non-contextual hidden variables (NCHV) which satisfy the following condition for Hilbert spaces whose dimensions are larger than two.

For any commuting observables A and B, the hidden variables (HV) are able to assign single values for $\langle A\rangle$ and $\langle B\rangle$ simultaneously for which $\langle aA+bB\rangle = a\langle A\rangle+b\langle B\rangle$ and $\langle AB\rangle = \langle A\rangle\langle B\rangle$ hold, with a and b being arbitrary real numbers.

The impossibility of the above-mentioned statement in the framework of QM was first proved by Bell [1] by reduction to absurdity, but direct proofs of the theorem have been searched [2–8]. It is known that the above-mentioned conditions for observables are translated into the following conditions [2, 9].

(i) For any unit vectors in the Hilbert space in question, values 0 or 1 are assigned *non-contextually*, i.e., independent from experimental situations. A valuation of unit vectors must also be independent of the phase of each unit vectors, i.e., the valuation is defined on rays in the Hilbert space.

(ii) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v_n}\}$ forms an orthonormal basis set for the whole quantum space in question, then just one and only one vector is assigned to value 1, and others are assigned to 0.

Then proofs of the BKS theorem are concentrated to show concrete examples consisting of finite vector sets which do not satisfy the above-mentioned conditions. Let us call sets of vectors which satisfy the above-mentioned conditions KS-colourable sets, and sets which do not satisfy them, KS-noncolourable. Examples of KS-noncolourable sets are found from the original 117-vector sets in three-dimensional real Hilbert space [2] to an 18-vector set in four-dimensional real Hilbert space [8], and recently Cabello, Estebaranz and García-Alcaine found a systematic method of constructing KS-noncolourable sets from known KS-noncolourable sets in lower dimensions [10]. CEG's method is simple and powerful enough but they also found that in the KS-noncolourable sets obtained by their method, some vectors are surplus, i.e., some vectors are reduced to yield smaller KS-noncolourable sets than the original ones. As noted by them, the KS-noncolourability of these vector sets was proved only by computer calculations. For example, they constructed a 31-vector KS-noncolourable set in five dimensions from the 18-vector set in four dimensions by using their method. Then they found that 29 of them are enough to be KS-noncolourable by a computer calculation.

In the present paper it will be shown that we can actually obtain CEG's results analytically with a slight modification of their method. For instance, the 29-vector set mentioned above is obtained automatically by the present method.

2. Modified CEG method

First we give the definition of KS-colouring. What we do is to colour each ray in a vector space V with either of two colours, black or white. Black corresponds to the valuation of 1 in the introduction and white to 0. We use vectors in place of rays for convenience unless confusion occurs, i.e., we use a vector $\mathbf{v}$ in the place of a ray ℓ when ℓ is spanned by $\mathbf{v}$. In the following we also assume vector spaces to be real just for simplicity.

Definition 2.1. *A set $\mathcal{S}$ formed by vectors of n-dimensional Euclid vector space V is called KS-colourable in V when:*

(1) At least one orthonormal basis set is included in $\mathcal{S}$, i.e., at least one mutually orthogonal n-vector set exists as a subset of $\mathcal{S}$.

(2) Two colours whether black or white are assigned to each element of $\mathcal{S}$ and if $\mathbf{v} \in \mathcal{S}$ is black, then all elements of $\mathcal{S}$ which are orthogonal to $\mathbf{v}$ are coloured white.

(3) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of V and is also a subset of $\mathcal{S}$, then precisely just one vector is coloured black and all other are coloured white.

On the other hand, a set $\mathcal{S}$ is called KS-*noncolourable* when conditions 2 and 3 for the definition of KS-colourable set cannot be simultaneously satisfied. The construction of a KS-noncolourable set in V proves the BKS theorem for the vector space V as mentioned in the introduction, and the main tool in the present paper is then stated as

Proposition 2.1. *Let V be an n-dimensional vector space and V_1, V_2 be m- and l-dimensional subspaces of V such that $V_1 + V_2 = V$, i.e., $m + l \geqslant n$. Suppose that the sets of unit vectors $\mathcal{S}_1 \subset V_1$ and $\mathcal{S}_2 \subset V_2$ are given and that $\mathcal{S}_i (i = 1, 2)$ contain at least one orthonormal basis* *set for V_i and also contain at least one orthonormal basis set for $V_j^{\perp}$ (here, $j \neq i$ and $W^{\perp}$ denotes the orthogonal complement of W in V). Then if $\mathcal{S}_1 \cup \mathcal{S}_2$ is KS-colourable in V, at least one of $\mathcal{S}_i$ is KS-colourable in V_i.*

Proof. By the assumption some elements of $\mathcal{S}_1$ or $\mathcal{S}_2$ are coloured black, and without loss of generality we may assume $\mathbf{u} \in \mathcal{S}_1 \subset V_1$ to be black. From the assumption we have $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-m}\} \subset \mathcal{S}_2$ as an orthonormal basis of $V_1^{\perp}$ and these elements are all coloured white because they are orthogonal to $\mathbf{u}$. Then we see that the limitation of the colouring of $\mathcal{S}_1 \cup \mathcal{S}_2$ to $\mathcal{S}_1$ gives the KS-colouring of $\mathcal{S}_1$ in V_1. Indeed, the colours of all vectors of $\mathcal{S}_1$ are already determined from the colouring of $\mathcal{S}_1 \cup \mathcal{S}_2$. And for any orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of V_1, from the assumption we have $\{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_{n-m}\}$ as an orthonormal basis of V, and from the KS-colourability of $\mathcal{S}_1 \cup \mathcal{S}_2$ we conclude that all $\mathbf{v}_i$ are coloured white and just one of $\mathbf{u}_i$ must be black, and all others white. Thus the proof is over. □

There are some different ways of constructing a KS-noncolourable set of vector space W from a set in lower-dimensional space V. For instance, if the dimension of W is a multiple of the dimension of V, i.e. $\dim W = k \dim V$ with k being an integer, we may regard W as a tensor product of V and $\mathbb{R}^k$. Then we can easily confirm that if $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a KS-noncolourable set in V, the set $\{\mathbf{v}_1 \otimes \mathbf{e}_1, \mathbf{v}_2 \otimes \mathbf{e}_1, \dots, \mathbf{v}_1 \otimes \mathbf{e}_k, \dots, \mathbf{v}_m \otimes \mathbf{e}_k\}$ becomes a KS non-colourable set in W where $\mathbf{e}_i$ are the standard bases of $\mathbb{R}^k$. If W is decomposed into a direct sum of vector spaces, e.g. $W = V_1 \oplus V_2$, where V_i have KS-noncolourable sets $\mathcal{S}_i$, then $\mathcal{S}_1 \cup \mathcal{S}_2$ becomes a KS-noncolourable set of W, as shown by Zimba and Penrose. [11]. So the construction of a KS-noncolourable set of W when $\dim W < 2 \dim V$ is the remaining question and CEG [10] gave an answer to it.

Suppose $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ with $n < m \leqslant 2n$. We now embed V in W in two different ways:

(i) $i_1 : \mathbf{a} \in V \to (\mathbf{a}, \mathbf{0}) \in W$, where $\mathbf{0}$ denotes the null vector in $\mathbb{R}^{m-n}$.
(ii) $i_2 : \mathbf{a} \to (\mathbf{0}, \mathbf{a})$.

What CEG has proved is that if $\mathcal{S}$ is a KS-non colourable set in V, then $i_1(\mathcal{S}) \cup i_2(\mathcal{S})$ plus some vectors, if needed, becomes a KS-noncolourable set in W.

Now let us examine the CEG-method. For a given KS-non colourable set $\mathcal{S}$ in the n-dimensional vector space V, our aim is to construct a KS-noncolourable set $\mathcal{T}$ in the larger space W from the set $\mathcal{S}$. As mentioned, CEG considered two kinds of embedding of V in W. Instead we shall regard the set $i_2(\mathcal{S})$ as an image of $i_1(\mathcal{S})$ under a certain orthogonal transformation T of W. As to the original CEG's treatment, T is written as

$$T = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{1}$$

where I on the upper right denotes the unit matrix of dimension $(m - n)$ and I on the lower left the unit matrix of dimension n. However, this choice of orthogonal transformation is not economical. Indeed, in some cases CEG needed to add some vectors to $\mathcal{T}$ to make a really KS-noncolourable set as mentioned. Instead of the original CEG's orthogonal transformation T in (1), we are able to choose a more economical one as follows.

Let W and V be m- and n-dimensional vector spaces with $n < m \leqslant 2n$ as above, and in addition we assume that V is a subspace of W, i.e., we choose an arbitrary embedding of V in W and fix it. We also assume that a KS-noncolourable set $\mathcal{S}$ in V is given. Let U be an arbitrary n-dimensional subspace of W. Then unless U equals V, $D = V \cap U$ becomes a $(2n - m)$-dimensional subspace and we have the following orthogonal decompositions:

$$V = V' \oplus D, \qquad U = U' \oplus D, \qquad W = V' \oplus D \oplus U', \tag{2}$$

where $V'(U')$ are $(m-n)$-dimensional subspaces.

Now let us choose D such that $\mathcal{S} \cap V'$ includes at least one orthonormal basis of $V' = U^{\perp}$. Then the following theorem holds.

Theorem 2.2. *Under the condition mentioned above, for any orthogonal transformation T of W which transforms V' onto $U' = V^{\perp}$ and acts as an identity on D, $\mathcal{S} \cup T\mathcal{S}$ becomes KS-noncolourable in W.*

Proof. From the assumption that $\mathcal{S} \cap V'$ includes at least one orthonormal basis of V', $T\mathcal{S} \cap U'$ includes at least one orthonormal basis of U' because $T\colon V' \to U'$ is isometric. Since $V' = U^{\perp}(U' = V^{\perp})$, we can apply proposition 2.1 and we readily see that $\mathcal{S} \cup T\mathcal{S}$ is KS-noncolourable because if it is KS-colourable, either $\mathcal{S}$ or $T\mathcal{S}$ must be KS-colourable from proposition 2.1 but it contradicts the assumption that $\mathcal{S}$ is KS-noncolourable. (Note that $T\mathcal{S}$ is KS-non colourable, too.)

As a concrete example of T, we can choose

$$T = \begin{pmatrix} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{pmatrix},$$

where I denotes the unit matrices of $(m-n)$ (off-diagonal part) and $(2n-m)$ (diagonal part) dimensions. Here we took the disjoint union of arbitrary orthonormal bases of V', D and U' as the orthonormal basis for the whole space W. Then the elements of $\mathcal{S}$ are written as

$$(\mathbf{a}, \mathbf{b}, \mathbf{0}),$$

and the elements of $T\mathcal{S}$ are written as

$$(\mathbf{0}, \mathbf{b}, \mathbf{a}).$$

□

In general, if an n-vector KS-noncolourable set in V is given, by finding a $(2n-m)$-dimensional subspace D of V satisfying the condition of theorem 2.2, we can construct a KS-noncolourable set $\mathcal{T}$ in W. The number of elements of $\mathcal{T}$ is given by $(n-k) \times 2 + k$, k being the number of elements of $\mathcal{S} \cap D$. Thus to find the smallest $\mathcal{T}$ starting from $\mathcal{S}$ is reduced to find a subspace D which contains the maximum number of elements of $\mathcal{S}$ while retaining the condition of theorem 2.2, i.e., $\mathcal{S}$ includes at least one orthonormal basis of the orthogonal complement space V' of D, in V.

Now let us apply the above-explained procedure to construct KS-noncolourable sets in five-, six- and seven-dimensional space starting from CEG's four-dimensional KS-noncolourable set which consists of 18 vectors in $\mathbb{R}^4$ [8].

Let us start from the $\mathbb{R}^5$ case. As described above, we assume that a four-dimensional space is embedded in a five-dimensional space arbitrarily and assume that CEG's 18-vector set $\mathcal{S}$ is given in the four-dimensional space. Here, the dimension of D is calculated to be $\dim D = 2 \times 4 - 5 = 3$. CEG's 18-vector set has a remarkable property, i.e., each vector has seven vectors orthogonal to it, meaning that these seven vectors span the orthogonal complement space to the first chosen vector in $\mathbb{R}^4$. That is, if we take one vector $\mathbf{u}$ in $\mathcal{S}$ and let D denote the three-dimensional subspace which is spanned by the elements of $\mathcal{S}$ which are orthogonal to $\mathbf{u}$, then $\mathbf{u}$ becomes a basis of V' in (2). Then by applying theorem 2.2, we find a $29 = (18-7) \times 2 + 7$-vector KS-noncolourable set in $\mathbb{R}^5$. It is apparent from the above-mentioned property of CEG's 18-vector set $\mathcal{S}$ that the maximum number of elements of $\mathcal{S}$ included in the three-dimensional subspace of $\mathbb{R}^4$ is seven, so this is the best result in the present method.

For $\mathbb{R}^6$, dim $D = 2$ and we find that three is the maximum number of vectors of $\mathcal{S}$ in D, and we obtain a 33 KS-noncolourable vector set which is larger than the record smallest set obtained by computer calculation, i.e., 31 of [10].

In the $\mathbb{R}^7$ case, dim $D = 1$ and of course only one vector is available, hence we have a 35-vector set. If we examine the properties of CEG's 18-vector set more closely, we are able to show that one vector can be subtracted from the 35-vector set and this coincides with the result of computer calculation by CEG [10]. The proof that we can subtract one vector is rather cumbersome, so it will be noted in the appendix.

3. Summary

By re-examining CEG's method of constructing KS-non colourable sets in higher-dimensional vector space from already known KS-noncolourable sets in lower dimension, we found a different way of constructing such sets. As the result, the number of vectors needed for constructing KS-noncolourable sets in higher dimension is reduced from the original method. Applying the new method to CEG's 18-vector set in $\mathbb{R}^4$, we found analytical proofs of the KS-noncolourability of a 29-vector set in $\mathbb{R}^5$ and the 34-vector set in $\mathbb{R}^7$. The KS-noncolourability of these sets is proved only by a computational method so far.

Appendix

Here we give the proof that the 34-vector set is enough to show the KS-theorem in $\mathbb{R}^7$. In figure 1(a), we show CEG's 18-ray (vector) set in $\mathbb{R}^4$ as a point set in $\mathbb{RP}^3$, i.e., we show the result of the gnomonic projection by regarding $(0, 0, 0, 1)$ as the 'north pole'. Then a ray spanned by a vector (a, b, c, d) is represented by the point $(a/d, b/d, c/d)$ in $\mathbb{R}^3$ when $d \neq 0$ and the ray $(a, b, c, 0)$ is represented by the point on the hyperplane at the infinity Π_∞, i.e. by the ray spanned by a vector (a, b, c) in $\mathbb{R}^3$. If two vectors $(\mathbf{a}_1, 1)$ and $(\mathbf{a}_2, 1)$, where $\mathbf{a}_{1,2} \in \mathbb{R}^3$, are orthogonal, we readily see that $(T\mathbf{a}_1, 1)$ and $(T\mathbf{a}_2, 1)$ are also orthogonal where T denotes any orthogonal transformation in $\mathbb{R}^3$. We also see that if $(\mathbf{a}_1, 1)$ and $(\mathbf{a}_2, 0)$ are orthogonal, $(T\mathbf{a}_1, 1)$ and $(T\mathbf{a}_2, 0)$ are orthogonal, and so on. Namely, if we find a tetrad, a four-ray set with each ray being orthogonal to others in $\mathbb{R}^4$ represented as a point set in $\mathbb{R}^3 \cup \Pi_\infty$, then any images of these points under rotation around the origin O also becomes a tetrad. In the following we make use of the above-mentioned fact and that point sets such as $XOAB$, $XZHI$, $EFGH$ form tetrads in $\mathbb{R}^4$. (see figure 1(b)) For instance, since $X = (1, 0, 0, 0)$, $Z = (0, 0, 1, 0)$, $H = (0, -1, 0, 1)$ and $I = (0, 1, 0, 1)$, we readily see that $XZHI$ form a tetrad. Once we confirm that these point sets form tetrads, we can find other tetrads in figure 1(a). For example, from the fact that $XOAB$ form a tetrad, we see $ZOCD$ form a tetrad, and so on.

Now let V_1 and V_2 be four-dimensional subspaces of $\mathbb{R}^7$. Then we have 18-vector sets which are similar to CEG's 18-vector set $\mathcal{S}_i$ in each space V_i, $i = 1, 2$. Here we can also assume that if we express the elements of $\mathcal{S}_i$ in the same manner as in figure 1(a), the point corresponding to O equals the ray in $V_1 \cap V_2$. Thus $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ consists of 35 points in $\mathbb{RP}^6$ and $\mathcal{S}$ is KS-noncolourable. In the following, we will prove that $\mathcal{S} - \{O\}$ is already KS-noncolourable.

First we note that we *do not have* KS-colouring of $\mathcal{S}_1 - \{O\}$ in $\mathbb{R}^4$ with the constraint of one of XAB *and* one of ZCD being black, where XAB and ZCD are mutually orthogonal 3-vector sets (triads) given in figure 1(a). Provided that the above-mentioned statement is proved, we are able to prove the main assertion as follows.

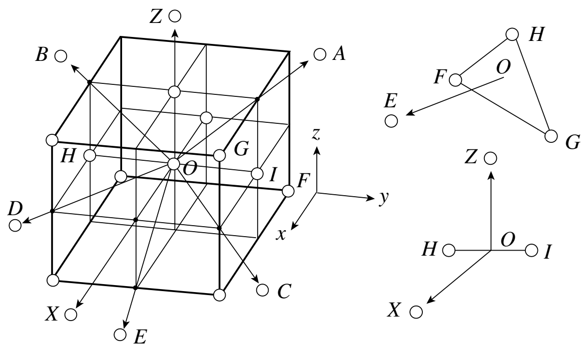

Figure 1. (*a*) 18-vector set of [10] illustrated as points in $\mathbb{RP}^3$. Points shown with arrows represent the intersection points of the lines shown by arrows and the hyperplane at infinity. The distance between the origin and each faces of the cube is unity so for example, F corresponds to $(-1, 1, -1, 1)$ in $\mathbb{R}^4$. (*b*) Two kinds of orthonormal basis sets (tetrads) appear in the proof. (O is not included in these tetrads.)

Suppose that $\mathcal{S} - \{O\}$ is KS-colourable, and such a colouring is obtained. Then without loss of generality, we can assume some points in $\mathcal{S}_1$ are black because $\mathcal{S}_1$ and $\mathcal{S}_2$ have exactly the same structure. Points in $\mathcal{S}_1 - \{O\}$ are divided into two subsets, namely $\mathcal{U}_1$, points which are orthogonal to O, and $\mathcal{W}_1$ consisting of the remaining points. From proposition 2.1, the limitation of a KS-colouring on $\mathcal{S} - \{O\}$ to $\mathcal{S}_1 - \{O\}$ gives a KS-colouring of the latter set in $\mathbb{R}^4$, where all points of $\mathcal{U}_1$ are white. Indeed, if there is a black point denoted as Y in $\mathcal{U}_1$, then, by the above-mentioned statement, we have at least one triad XAB or ZCD, with its all elements coloured white. Since all points of $\mathcal{S}_2$ are orthogonal to Y they are all coloured white. (Note that using the notations of the main text, Y is an element of V_1' and $\mathcal{S}_2 \subset V_2$ with $V_1' \perp V_2$.) If we choose any tetrad in $\mathcal{S}_2$ then the union of a white coloured triad just mentioned above and the chosen tetrad form seven mutually orthogonal vector set with all white points contradicting to the KS-colouring rule. Thus all black points should be in $\mathcal{W}_1$ but this will also lead us to the contradiction as shown below. Instead of the first given colouring of $\mathcal{S} - \{O\}$, we obtain a modified new KS-colouring as follows. Let us divide $\mathcal{S}_2$ into two subsets $\mathcal{U}_2$ and $\mathcal{W}_2$ exactly the same way which we did for $\mathcal{S}_1$ and colour all points of $\mathcal{U}_2$ white and colour points of $\mathcal{W}_2$ in accordance with the colour of corresponding points of $\mathcal{W}_1$ are black or white. Such a colouring is easily proved to be a KS-colouring on $\mathcal{S} - \{O\}$ using the fact that all points of $\mathcal{W}_1$ and $\mathcal{W}_2$ are not mutually orthogonal and the fact that we cannot construct seven mutually orthogonal seven-vector set from $\mathcal{U}_1 \cup \mathcal{U}_2$. Since O is not orthogonal to $\mathcal{W}_1 \cup \mathcal{W}_2$, adding O as a black point to $\mathcal{S} - \{O\}$ gives a *KS-colouring on* $\mathcal{S}$, which is a contradiction. In conclusion, we find that $\mathcal{S} - \{O\}$ itself must be KS-noncolourable.

Now what remains is to show the impossibility of a KS-colouring of $\mathcal{S}_1 - \{O\}$ in $\mathbb{RP}^3$ with one of XAB and one of ZCD being black. From figure 1(*a*), we readily see that $\mathcal{S}_1$ has reflection symmetry with respect to the zx plane, so what we have to prove is reduced to showing that there is no KS-colouring of $\mathcal{S}_1 - \{O\}$ with (i) X and C being black, (ii) Z and A being black, (iii) A and C being black or (iv) A and D being black.

Proofs for these four cases are done in similar manner so that only a proof for the case (i) is given here and others are left to readers.

Now let us assume X and C are both black in a colouring of $\mathcal{S}_1 - \{O\}$. We introduce an additional assumption that the point E in figure 2(*a*) is also black and then we have a

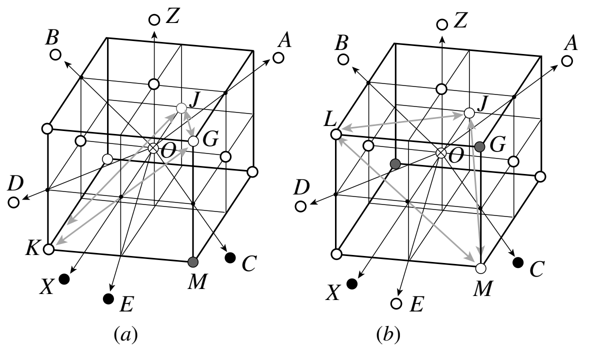

Figure 2. Eye guide for proving that KS-noncolourability with both X and C being black at the same time with the additional assumption where E is assumed to be black in (a) while in (b), white.

contradiction. From the assumption that X and C are black, we find that the thick lined circles in figure 2(a) are all white. By the additional assumption that E is black, the point G and its diagonal vertex become white because they are orthogonal to E. Then by applying the KS-colouring rule—just one of the elements of each tetrad in $\mathcal{S} - \{O\}$ must be coloured black—to the tetrad which includes the point D, the point M becomes black. As a result, we obtain the colouring of the remaining point J as white because J is orthogonal to M. However, the obtained colouring does not satisfy the KS-colouring rule because the members of the tetrad $BGJK$ are all coloured white.

Now, it is sufficient to show colouring with X and C being black and E white is impossible (see figure 2(b)). From the assumption that E is white, we see G and its diagonal vertex must be black considering the two-tetrads involving E. Then the colouring of the remaining points is completed because they are orthogonal to these circles in the figure. And, as is before, we are forced to have the tetrad $AJLM$ with all white colour.

Using the same technique we can show KS-noncolourability for cases (ii)–(iv); we succeed in proving the KS-noncolourability of the 34-vector set in $\mathbb{R}^7$.

References

[1] Bell J S 1966 *Rev. Mod. Phys.* **38** 447
[2] Kochen S and Specker E P 1967 *J. Math. Mech.* **17** 59
[3] Mermin N D 1990 *Phys. Rev. Lett.* **65** 3373
[4] Reported in, Peres A 1993 *Quantum Theory: Concepts and Method* (Dordrecht: Kluwer) p 114
[5] Peres A 1991 *J. Phys. A: Math. Gen.* **24** L175
[6] Kernaghan M *J. Phys. A: Math. Gen.* **27** L829
[7] Kernaghan M and Peres A 1995 *Phys. Lett.* A **198** 1
[8] Cabello A, Estebaranz J M and García-Alcaine G 1996 *Phys. Lett.* A **212** 183
[9] Mermin N D 1993 *Rev. Mod. Phys.* **65** 803
[10] Cabello A, Estebaranzn J M and García-Alcaine G 2005 *Phys. Lett.* A **339** 425
[11] Zimba J and Penrose R 1993 *Stud. Hist. Phil. Sci.* **24** 697